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A Transition in a Noisy Linear System Driven by a Periodic Signal

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We consider a one-dimension linear random walk between two trapping points in which the transition probabilities vary periodically in time. An earlier analysis of this system showed that the mean time to trapping of a particle in this system exhibits a minimum when considered as a function of frequency. In this note we show that this parameter makes a transition in behavior from a monotonic decrease with increasing amplitude of the periodic term to a monotonic increase with this parameter depending on the frequency. A physical argument is suggested to explain this behavior. Confirmation of this crossover can also be derived from a diffusion model.

KEY WORDS: Stochastic resonance; first passage times.

A considerable literature has recently developed that is devoted to the analysis of the interaction of noise and periodic forcing functions in the framework of nonlinear dynamical systems. The field of stochastic resonance, for example, is devoted to an examination of the consequences of just such interactions.(1) While it is not too surprising that there are interesting effects for nonlinear systems, there are also a number of partially unexpected features that emerge from the analysis of interactions of noise and periodic forcing functions even for linear systems. Some of the more unexpected effects produced by an oscillatory field can be of more than academic interest, since pulse-field electrophoresis plays a prominent role in the armamentarium of the biochemist,(2,3) particularly in the area of protein separations. Some time ago Fletcher *et al.*(4) studied the effects of

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the frequency of an oscillating field on the average survival time of a one-dimensional random walker or Brownian particle that diffuses on a line connecting two traps. The emphasis in the cited paper was on the effects of frequency on the mean first passage time to trapping (MFPT). In particular, it was shown that the MFPT, considered as a function of the frequency of the sinusoidal forcing time ω, both for the random walk in discrete time and for the diffusion process, exhibits a minimum as the frequency ω is varied. This resonance effect can be explained in fairly straightforward terms. In this note we examine the MFPT not as a function of the frequency, but rather as a function of the amplitude of the forcing field, a parameter that we denote by $\varepsilon/2$. The MFPT occurs as an important parameter in a theoretical analysis of the motion of a DNA molecule in the biased reptation model.[5-7] We show that in our simplified model the MFPT exhibits a type of transition in that for some frequencies the MFPT increases with increasing ε, for others it is a nonmonotonic function of ε, and for still others it is a decreasing function of the amplitude of the bias parameter. We also present a physical picture to support this apparently unintuitive behavior.

In order to demonstrate the behavior of the mean first passage time as a function of amplitude, we first simulated a random walk on a discrete lattice. Let X_n be the position of the random walker at step n,

$$X_{n+1} = X_n + v_n \tag{1}$$

where the $\{v_n\}$ are independent random variables whose properties are specified by the prescription

$$v_n = \pm 1 \quad \text{with probability} \quad \tfrac{1}{2}[1 \pm \varepsilon \cos(\omega n)] \tag{2}$$

where $0 < \varepsilon \leqslant 1$.

The traps are assumed located at sites 0 and $2a$, and the initial location of the random walker is at site a. Let the probability that the random walker is at site j at step n be denoted by $p_n(j)$. This function satisfies the evolution equation

$$p_{n+1}(j) = \tfrac{1}{2}[1 + \varepsilon \cos(\omega n)]\, p_n(j-1) + \tfrac{1}{2}[1 - \varepsilon \cos(\omega n)]\, p_n(j+1) \tag{3}$$

which must be solved subject to the initial and boundary conditions

$$p_0(j) = \delta_{j,a} \qquad \text{and} \qquad p_n(0) = p_n(2a) = 0 \tag{4}$$

While the combination of Eqs. (3) and (4) is solvable in closed form at zero frequency, a numerical solution of the recursion relation is required for time-dependent transition probabilities. When the $p_n(j)$ are known, the MFPT, $\langle n(\varepsilon, \omega)\rangle$, can be calculated as

$$\langle n(\varepsilon, \omega)\rangle = \sum_{j=0}^{2a} \sum_{n=0}^{\infty} p_n(j) \tag{5}$$

Notice that one cannot use a formalism based on an adjoint operator to calculate $\langle n(\varepsilon, \omega)\rangle$, because the coefficients in Eq. (3) depend on n.

Specific results for $\langle n(\varepsilon, \omega)\rangle$ regarded as a function of ε are plotted in Fig. 1 for the value $a = 25$. The three curves that are shown indicate a transition in behavior from the strict decrease in the low-ω regime to values of ω which lead to nonmonotonic behavior as the amplitude of the periodic term is increased, and a final transition to a regime in which the function $\langle n(\varepsilon, \omega)\rangle$ increases monotonically with ε. A qualitative explanation of the results shown in Fig. 1 is that as ε increases, the random walker is initially pushed toward the boundary at $2a$. Because the cosine term eventually reverses sign, those particles that survive tend to move toward the boundary at 0, their average position during the cycle passing through the initial position. However, since they are now further from $j = 2a$ than if they had been at their initial point, they are less likely to reach that trapping point in the following cycle. If the amplitude ε is sufficiently great, the random walker which started at lattice site a will find itself at some lattice site to the left of $j = a$ at the end of the second cycle. Hence, when the third cycle starts, it will be harder to reach the trapping point at $2a$ than it was at $n = 0$. This type of behavior will then continue and may be regarded as

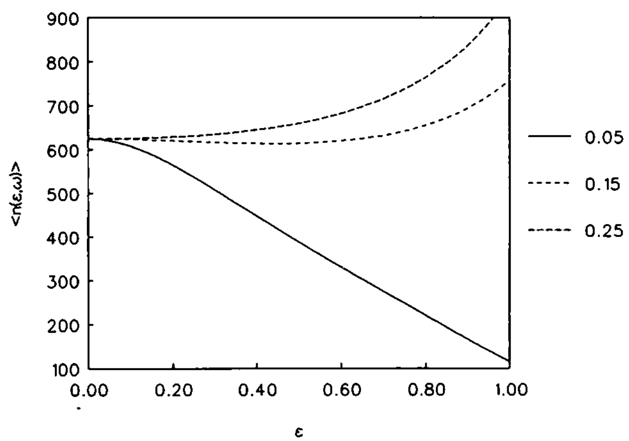

Fig. 1. Curves of $\langle n(\varepsilon, \omega)\rangle$ plotted as a function of the amplitude ε for the value $a = 25$ and three values of the driving frequency ω. The curve drawn for $\omega = 0.15$ exhibits a very slight dip before it begins to increase, while the other two curves show monotonic behavior.

a feedback mechanism, suggesting the increase in $\langle n(\varepsilon, \omega)\rangle$ observed for $\omega = 0.25$ in Fig. 1. Similar results are valid for other values of interval length than the one chosen to generate the figure.

We have demonstrated the existence of three types of qualitative behavior of $\langle n(\varepsilon, \omega)\rangle$ regarded as a function of ε. The three regimes depend on the value of ω. At sufficiently low values of ω (a regime specified by $\omega < \omega_1$) $\langle n(\varepsilon, \omega)\rangle$ is a strictly decreasing function of ε, while in the intermediate regime ($\omega_1 \leqslant \omega < \omega_2$) the function $\langle n(\varepsilon, \omega)\rangle$ has a minimum when it is considered as a function of ε. Finally, in the high-frequency region, $\omega > \omega_2$, $\langle n(\varepsilon, \omega)\rangle$ increases monotonically with ε. The critical parameters ω_1 and ω_2 have been calculated as a function of the length of the interval. Some results of this calculation are shown in Table I. It is clear that the critical frequencies must decrease as the length of the interval increases, since the bias that tends to drive the particle toward the trap during the first cycle must be effective for a long enough time for there to be a significant probability that the particle will actually reach the trap during that cycle. Since the average bias during the first cycle in which $\cos(\omega t) > 0$ is proportional to $1/\omega$, we expect that ω_1 should be approximately proportional to $1/a$ at sufficiently large values of a. These predictions are in rough agreement with the data in the table for $a = 50$ and 100. Similarly, we expect that ω_2 is approximately proportional to $1/a$, which again is approximately confirmed by the data at the largest values of a. While we have only demonstrated a kind of phase transition in the case of the MFPT, we anticipate that such effects will occur in higher moments of the first passage time as well.

It is possible, by considering the limiting cases $\varepsilon = 0$ and $\varepsilon = 1$, to see that there must be a transition in the behavior of $\langle n(\omega, \varepsilon)\rangle$ regarded as a function of ω. However, the discrete-time formulation leads to rather complicated equations, while the continuum analog leads to results that are somewhat easier to analyze. This analog consists of a diffusion equation augmented by a sinusoidal velocity term:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} - v\cos(\omega t)\frac{\partial p}{\partial x} \tag{6}$$

Table I

	$a = 10$	$a = 25$	$a = 50$	$a = 100$
ω_1	0.185	0.086	0.048	0.028
ω_2	0.32	0.195	0.120	0.08

where D is a diffusion constant and v a constant velocity. The initial and boundary conditions that $p(x, t)$ must satisfy are

$$p(x, 0) = \delta(x - a), \qquad p(0, t) = p(2a, t) = 0 \tag{7}$$

It is known that finding a solution to Eq. (6) in the presence of trapping sites poses a notoriously difficult problem.[8] However, we can consider the two limiting cases of interest in which the amplitude of the forcing field goes to zero, and the second case in which diffusion can be regarded as being negligible in comparison with the biased motion.

In pursuit of this goal it is convenient to rephrase Eq. (6) in terms of the dimensionless variables

$$\varepsilon = \frac{2av}{D}, \qquad y = \frac{x}{2a}, \qquad \tau = \frac{Dt}{4a^2}, \qquad \nu = \frac{4a^2\omega}{D} \tag{8}$$

to enable us to specify the large or small parameters that characterize the process. Thus, the parameter ε is the dimensionless amplitude of the forcing term. Equation (6) can now be rewritten in terms of these parameters as

$$\frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial y^2} - \varepsilon \cos(\nu\tau)\frac{\partial p}{\partial y} \tag{9}$$

with the boundary conditions $p(0, \tau) = p(1, \tau) = 0$ and the initial condition $p(y, 0) = \delta(y - 1/2)$. The two limits to be considered are $\varepsilon = 0$, which corresponds to a pure diffusion process, and $\varepsilon \gg 1$. In the second of these cases we argue that the diffusive term can be dropped. The resulting equation then corresponds to the strictly deterministic process

$$\dot{y} = \varepsilon \cos(\nu\tau) \tag{10}$$

Dropping the diffusive term can be justified provided that $\varepsilon \cos(\nu\tau) \gg 1$, which means that either the particle reaches the boundary during the first cycle of the deterministic motion or does not do so at all.

The dimensionless mean first passage time for pure diffusion ($\varepsilon = 0$) is readily found to be $\langle \tau \rangle_{\text{diff}} = 1/8$, while for the deterministic process one finds

$$\langle \tau \rangle_{\text{det}} = \frac{1}{\nu} \sin^{-1}\left(\frac{\nu}{2\varepsilon}\right) \tag{11}$$

or, in the original dimensioned variables,

$$\langle t \rangle_{\text{diff}} = \frac{a^2}{2D}; \qquad \langle t \rangle_{\text{det}} = \frac{1}{\omega} \sin^{-1}\left(\frac{\omega a}{v}\right) \tag{12}$$

A comparison of these expressions shows that when ω is held fixed and the parameter a is varied it is possible for $\langle t\rangle_{\mathrm{diff}}/\langle t\rangle_{\mathrm{det}}$ to be either greater or less than 1, depending on the frequency ω, a conclusion which agrees with the extreme values in our simulated random walk data. In this picture the frequency at which a crossover occurs between the two limiting behaviors ω_c is the solution to

$$\operatorname{sinc}\left(\frac{a^2\omega_c}{2D}\right)=\frac{2D}{av} \tag{13}$$

in which $\operatorname{sinc}(x)\equiv\sin(x)/x$. Since $\operatorname{sinc}^2(x)\leqslant 1$ Eq. (13) implies that a crossover can only occur provided that v satisfies $v>2D/a$.

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